Tutorial 4

1. Let $c = a + bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What additional restriction must be placed on the constant c so that values of $|i^c|$ are all the same?

Solution. Recall that for $z \neq 0$, we have

$$z^c := \exp(c \log z),$$

where $\log z$ denotes the multiple-valued logarithmic function defined by

$$\log z := \ln |z| + i \arg z.$$

Therefore, by definition, we can write

$$\begin{aligned} i^c = \exp(c \log i) &= \exp\left\{(a + bi)\left[\ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right)\right]\right\} \\ &= \exp\left[-b\left(\frac{\pi}{2} + 2n\pi\right) + ia\left(\frac{\pi}{2} + 2n\pi\right)\right] \quad (n = 0, \pm 1, \pm 2, \ldots). \end{aligned}$$

Remark. The assumption that $c \neq 0, \pm 1, \pm 2, \ldots$ ensures i^c is multiple-valued.

It follows that

$$|i^c| = \exp\left[-b\left(\frac{\pi}{2} + 2n\pi\right)\right] \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Clearly, in order for $|i^c|$ to have a single value, we must place the restriction that $b = 0$, or c is real.

2. With the aid of the following expressions

$$\begin{aligned} \sin z &= \sin x \cosh y + i \cos x \sinh y, \\ \cos z &= \cos x \cosh y - i \sin x \sinh y, \end{aligned}$$

show that

(a) $\overline{\cos(iz)} = \cos(i\overline{z})$ for all z.

(b) $\overline{\sin(iz)} = \sin(i\overline{z})$ if and only if $z = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

Solution. Let $z = x + iy$.

(a)

$$\begin{aligned} \overline{\cos(iz)} = \overline{\cos(-y + ix)} &= \overline{\cos(-y)\cosh x - i \sin(-y) \sinh x} \\ &= \overline{\cos y \cosh x + i \sin y \sinh x} \\ &= \cos y \cosh x - i \sin y \sinh x \\ &= \cos(y + ix) \\ &= \cos(i\overline{z}). \end{aligned}$$

(b) Firstly, we have

$$\begin{aligned} \overline{\sin(iz)} = \overline{\sin(-y + ix)} &= \overline{\sin(-y)\cosh x + i \cos(-y) \sinh x} \\ &= \overline{-\sin y \cosh x + i \cos y \sinh x} \\ &= -\sin y \cosh x - i \cos y \sinh x \end{aligned}$$

and

$$\sin(i\bar{z}) = \sin(y + ix) = \sin y \cosh x + i \cos y \sinh x.$$

Equating them yields

$$-\sin y \cosh x = \sin y \cosh x,$$
$$-\cos y \sinh x = \cos y \sinh x.$$

Since the quantity

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

is never zero, it follows from the first equation that $\sin y = 0$ i.e.

$$y = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Now, since $\cos y = \cos n\pi = \pm 1$ is never zero, it follows from the second equation that $\sinh x = 0$ i.e.

$$x = 0.$$

Therefore, we conclude that $z = x + iy = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

3. With the aid of the following expression

$$\cos z = \cos x \cosh y - i \sin x \sinh y,$$

find the roots of the equation $\cos z = 2$.

Solution. Let $z = x + iy$. By the formula, we need to solve

$$\cos x \cosh y - i \sin x \sinh y = 2$$

for x and y i.e.

$$\cos x \cosh y = 2,$$
$$\sin x \sinh y = 0.$$

Observe that y cannot be zero, otherwise it would follow from the first equation that $\cos x = 2$ which is impossible. In particular, we have $\sinh y$ is never zero and it would follow from the second equation that $\sin x = 0$ i.e.

$$x = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

However, from the first equation, we have $(\pm 1) \cosh y = 2$. Since $\cosh y$ is always positive, it follows that n must be even i.e.

$$x = 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Also, we have $\cosh y = 2$, or

$$y = \cosh^{-1} 2.$$

Thus, the roots are

$$z = 2k\pi + i \cosh^{-1} 2 \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Furthermore, solving $\cosh y = 2$ for y yields

$$\begin{aligned} & e^y + e^{-y} = 4 \\ \iff & (e^y)^2 - 4(e^y) + 1 = 0 \\ \iff & e^y = 2 \pm \sqrt{3} \\ \iff & y = \ln(2 \pm \sqrt{3}). \end{aligned}$$

Note that

$$\ln(2 - \sqrt{3}) = -\ln(2 + \sqrt{3}),$$

so alternatively we may write the roots as

$$z = 2k\pi \pm i \ln(2 + \sqrt{3}) \quad (k = 0, \pm 1, \pm 2, \ldots).$$